

# The effect of temperature on viscous friction and the performance of a Brownian heat engine

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We explore the transport features of a Brownian particle that walking in a periodic ratchet potential that is coupled with a spatially varying temperature background. Since the viscous friction of the medium decreases as the temperature of the medium increases, any reasonable exploration regarding the thermodynamic features of the Brownian engine should take into account the role of temperature on the viscosity of the fluid. In this work, we study this effect of temperature by considering a viscous friction that decreases exponentially as the background temperature increases. Our result depicts that the Brownian particle exhibits a fast unidirectional motion when the viscous friction is temperature dependent than that of constant viscous friction. Moreover the efficiency of this motor is considerably enhanced when the viscous friction is temperature dependent. On the hand, the motor exhibits a higher performance of the refrigerator when the viscose friction is taken to be constant.

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## I. INTRODUCTION

Over the past decade or two there has been a great interest in the study of noise-induced transport features of micron and nanometer sized particles. This was motivated not only for a better understanding of the nonequilibrium statistical physics of such systems but also due to the desire to construct artificial tiny motors that operate at the microscopic or nanoscopic levels [1, 2]. Several studies have shown that the dynamics of these particles exhibit a unidirectional motion when they are exposed to a bistable potential that is subjected to a spatial or temporal symmetry breaking fields such as inhomogeneous temperature background [3–10]. In particular when the bistable potential is exposed to a spatially varying temperature, the particles will have a fast unidirectional motion where the intensity of the current rectification depends strongly on the strength of background temperature and the potential barrier height. While undergoing a rectified motion along the reaction coordinate, these particles perform a useful work at the expense of the heat taken out from the hotter reservoir thus act as a Brownian heat engines. For detailed practical applications as well as the characteristics, and working principle of these classical Brownian motors, the reader is encouraged to refer to the works of Peter Hänggi and his collaborators [11, 12].

Several studies have been also conducted to understand the factors that affect performance of a Brownian engine that is driven by a spatially varying tempera-

ture. These studies have explored the different operational regimes of the engine both at the quasistatic limit and when the engines operate at finite time interval [13–20]. In line with these studies, we have explored in our previous work [21] the effect of thermal inhomogeneity on the performance of such heat engines by considering a Brownian particle in a ratchet potential that moves through a highly viscous medium. In the model considered in [21], the Brownian particle is driven by the thermal kick it receives from a linearly decreasing background temperature. The study showed that even though the energy transfer due to kinetic energy is neglected, Carnot efficiency cannot be achieved at quasistatic limit. At quasistatic limit, the efficiency for such a Brownian heat engine approaches the efficiency ( $\eta$ ) of an endoreversible heat engine; *i. e.*  $\eta = 1 - \sqrt{\frac{T_h}{T_c}}$  [22]. More recently, by considering a Brownian motor that operates between two different heat baths, we have also explored both the nonequilibrium steady state (NESS) and short time behavior of the engine [23]. This investigation studied the thermodynamic feature of the engine for both the isothermal case with a load and nonisothermal case with and without a load.

So far most of the studies of Brownian heat engines considered only temperature invariance viscous friction. However, it is well know that the viscosities of different media tend to depend on the intensity of the background temperature [24]. In liquid or glassy meadium viscosity tend to decrease when the intensity of the background temperature increases. This is

due to the fact that an increase in temperature of the medium brings more agitation to the molecules in the medium, and hence increases their speed. This speedy motion of the molecules creates a reduction in interaction time between neighboring molecules. In turn, at macroscopic level, there will be a reduction in the intermolecular force, and hence reduced viscosity of the fluid. Consequently, as the temperature of the viscous medium decreases, the viscous friction in the medium decreases. Meanwhile for a position dependent temperature along the reaction coordinate, the viscous friction is also spatially dependent. In such case we want to stress that the effect of temperature on the particles mobility and performance of the motor will be twofold. First, it directly assists the particles to surmount the potential barrier; *i. e.* particles jump the potential barrier at the expenses of the thermal kicks. Second, when temperature increases, the viscous friction gets attenuated and particles experiences a reduced inertial effect, which in turn increases the particles mobility and efficiency.

In this paper, we address the role of temperature dependent viscous friction on the Brownian heat engine by considering an exponential temperature dependence of friction,  $\gamma(x) = Be^{-AT(x)}$ , as proposed originally by Reynolds [25]. Our analysis shows that whether  $\gamma$  is temperature dependent or not, at quasistatic limit one always gets a Carnot efficiency and a Carnot refrigerator so long as the heat exchange via kinetic energy is omitted. However, when the heat exchange via the kinetic energy is included, it will be impossible to attain the Carnot efficiency or Carnot refrigerator even at quasistatic limit. Moreover, far from quasistatic limit, the engine exhibits an enhanced performance when the viscous friction is taken to be temperature dependent.

The rest of the paper is organized as follows. In section II, we present our model for the system. In section III, by considering a viscous friction that decreases exponentially with temperature, we explore the dependence of mobility, efficiency and performance of the refrigerator on the the model parameters. Finally we give summary and conclusion of the paper in Section IV.

## II. THE MODEL

We consider a Brownian particle that rattles in a one dimensional piecewise linear bistable potential with an external load; *i. e.*  $U(x) = U_s(x) + fx$ , where the ratchet potential  $U_s(x)$  is described by

$$U_s(x) = \begin{cases} 2U_0 \left( \frac{x}{L_0} \right), & \text{if } 0 \leq x \leq \frac{L_0}{2}; \\ 2U_0 \left( 1 - \frac{x}{L_0} \right), & \text{if } \frac{L_0}{2} \leq x \leq L_0. \end{cases} \quad (1)$$

Here  $U_0$  and  $L_0$  denote the barrier height and the width of the ratchet potential, respectively, and  $f$  is the strength of the load. The potential exhibit its maximum value  $U_0$  at  $x = \frac{L_0}{2}$  and its minima at  $x = 0$  and  $x = L_0$ . The background temperature in the system is taken to be spatially varying hot and cold regions where

$$T(x) = \begin{cases} T_h, & \text{if } 0 \leq x \leq \frac{L_0}{2}; \\ T_c, & \text{if } \frac{L_0}{2} \leq x \leq L_0 \end{cases} \quad (2)$$

as shown in Fig. 1. In this model both the potential  $U_s(x)$  and the background temperature  $T(x)$  are assumed to be periodic with period  $L_0$ ; *i. e.*  $U_s(x + L_0) = U_s(x)$  and  $T(x + L_0) = T(x)$ .

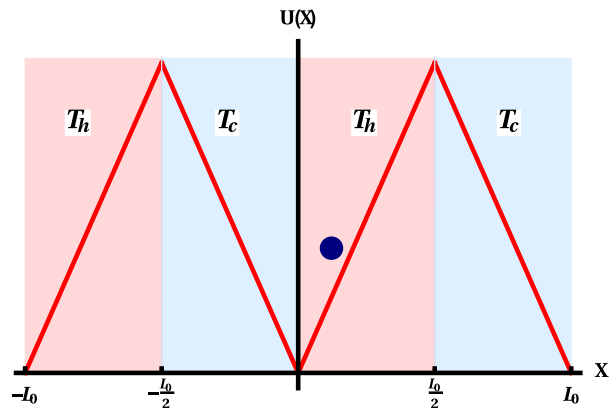


FIG. 1: (Color online) Schematic diagram for a particle in a piecewise linear bistable potential in the absence of an external load. The potential exhibits a potential maxima  $U_0$  at  $x = \frac{L_0}{2}$ . The potential minima is located at  $x = 0$  and  $x = L_0$ . Due to the thermal background kicks, the particle ultimately attains a steady state velocity as long as a temperature difference between the hot and cold reservoirs is retained.

For a Brownian particle that is arranged to undergo a random walk in a highly viscous medium, regardless of the magnitude of the external bias force, the particle has very little chance to accelerate along the medium. Hence one can safely neglect the inertial contribution to the Langevin equation and the dynamics of the particle under the influence of external potential  $U(x)$  is governed by

$$\gamma(x) \frac{dx}{dt} = -\frac{\partial U(x)}{\partial x} + \sqrt{2k_B \gamma(x) T(x)} \xi(t), \quad (3)$$

where  $\gamma$  is the viscous friction, and  $k_B$  is the Boltzmann's constant. As mentioned earlier, in this work the viscous friction is taken to have an exponential temperature dependence of the form  $\gamma(x) = Be^{-AT(x)}$  where  $A$  and  $B$  are constants. Since  $T(x)$  is spatially

variable we have,

$$\gamma(x) = \begin{cases} Be^{-AT_h}, & \text{if } 0 \leq x \leq \frac{L_0}{2}; \\ Be^{-AT_c}, & \text{if } \frac{L_0}{2} \leq x \leq L_0. \end{cases} \quad (4)$$

In this system the random noise  $\xi(t)$  is assumed to be Gaussian and white, satisfying  $\langle \xi(t) \rangle = 0$  and  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ . Hereafter we will work on the units where  $k_B$  and  $B$  are unity.

In the high friction limit, the dynamics of the Brownian particle is governed by [13]

$$\frac{\partial P(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[ \gamma^{-1} \left( U'(x)P(x, t) + \frac{\partial}{\partial x} (T(x)P(x, t)) \right) \right] \quad (5)$$

where  $P(x, t)$  is the probability density of finding the particle at position  $x$  at and at time  $t$ , and  $U'(x) = \frac{d}{dx}U$ . At steady state the current is given by  $J(x) = -[U'(x)P_s(x) + \frac{\partial}{\partial x} [T(x)P_s(x)]]$ . For periodic boundary condition,  $P_s(x + L_0) = P_s(x)$ , the corresponding steady state current  $J(x)$  can be evaluated exactly using the same approach shown in [13]. After some algebra, the closed form expression for the steady state current is given as

$$J = -\frac{\varsigma_1}{\varsigma_2\varsigma_3 + (\varsigma_4 + \varsigma_5)\varsigma_1} \quad (6)$$

where the expressions for  $\varsigma_1$ ,  $\varsigma_2$ ,  $\varsigma_3$ , and  $\varsigma_4$  are given by

$$\begin{aligned} \varsigma_1 &= -1 + e^{\frac{L_0(f - \frac{2U_0}{L_0})}{2T_c} + \frac{L_0(f + \frac{2U_0}{L_0})}{2T_h}} \quad (7) \\ \varsigma_2 &= \frac{e^{-\frac{fL_0(T_c + T_h) + 2U_0}{2T_h}} \left( e^{\frac{fL_0}{2T_c}} - e^{\frac{U_0}{T_c}} \right) L_0}{fL_0 - 2U_0} \\ &\quad - \frac{\left( e^{-\frac{fL_0 + 2U_0}{2T_h}} - 1 \right) L_0}{fL_0 + 2U_0} \\ \varsigma_3 &= \frac{e^{-AT_c - \frac{U_0}{T_c} + \frac{fL_0 + 2U_0}{2T_h}} \left( e^{\frac{fL_0}{2T_c}} - e^{\frac{U_0}{T_c}} \right) L_0 T_c}{fL_0 - 2U_0} + \\ &\quad \frac{e^{-AT_h} \left( e^{\frac{fL_0 + 2U_0}{2T_h}} - 1 \right) L_0 T_h}{fL_0 + 2U_0} \\ \varsigma_4 &= e^{-AT_h} L_0^2 \left( \frac{fL_0 + 2((-1 + e^{-\frac{fL_0 + 2U_0}{2T_h}})T_h + U_0)}{2(fL_0 + 2U_0)^2} \right) \end{aligned}$$

and  $\varsigma_5 = L_0^2(t_1 + t_2 + t_3 + t_4)$ . Here  $t_1$ ,  $t_2$ ,  $t_3$ , and  $t_4$  are given by

$$\begin{aligned} t_1 &= \frac{e^{-AT_c}}{2fL_0 - 4U_0} \quad (8) \\ t_2 &= \frac{e^{-AT_c}(1 - e^{-\frac{fL_0 - 2U_0}{2T_c}})T_c}{(fL_0 - 2U_0)^2} \end{aligned}$$

$$\begin{aligned} t_3 &= \frac{e^{-AT_h}(1 - e^{-\frac{fL_0 - 2U_0}{2T_c}})T_h}{(f^2L_0^2 - 4U_0^2)} \\ t_4 &= \frac{e^{-AT_h - \frac{fL_0(T_c + T_h)}{2T_c} - \frac{U_0}{T_h}} (-e^{\frac{fL_0}{2T_c}} + e^{\frac{U_0}{T_c}})T_h}{(f^2L_0^2 - 4U_0^2)}. \end{aligned}$$

In the absence of external load, *i. e.*  $f = 0$ , Eq. (6) reduces to

$$J = \frac{4(z_1 z_2)U_0^2}{L_0^2(\psi_1 + \psi_2 + \frac{2(z_1 T_c + z_2 T_h)(1 + e^{\frac{U_0}{T_c}}(-2 + e^{\frac{U_0}{T_c}}))}{e^{\frac{U_0}{T_c}} - e^{\frac{U_0}{T_h}}})} \quad (9)$$

where  $\psi_1 = -2z_1 T_c - 2z_2 T_h$ ,  $\psi_2 = -z_2 U_0 + z_1 U_0$ ,  $z_1 = e^{AT_h}$  and  $z_2 = e^{AT_c}$ .

When  $A = 0$ , the model will be reduced to a constant viscous friction  $\gamma$ , the case that was studied before and Eq. (9) reproduces the result of [13],

$$J^C = \frac{2U_0^2}{(T_h + T_c)} \left[ \frac{1}{e^{\frac{U_0}{T_h}} - 1} - \frac{1}{e^{\frac{U_0}{T_c}} - 1} \right]. \quad (10)$$

For a small barrier height (small  $U_0$ ), the steady state current converges to

$$J \approx \frac{4U_0(T_h - T_c)e^{A(T_h + T_c)}}{L_0^2(T_h + T_c)(e^{AT_c} + e^{AT_h})}. \quad (11)$$

On the other hand for large  $U_0$  the current can be approximated as

$$J \approx \frac{2U_0^2 e^{A(T_h + T_c) - \frac{U_0}{T_h}}}{L_0^2(e^{AT_c} T_h + e^{AT_h} T_c)}. \quad (12)$$

The drift velocity  $V$  of the Brownian particle is associated to the steady state current  $J$  via  $V = L_0 J$ .

As discussed before, a non-vanishing particle current  $J$  can be obtained as long as a distinct temperature difference between the hot and cold reservoirs is maintained even in the absence of a load. However, the external load is necessary to get a non-vanishing current for isothermal symmetric ratchet. Whether isothermal or not, the direction of the current is always dictated by the load. In the regime where  $J > 0$ , the model acts as a heat engine and in this case, in one cycle, a minimum  $(U_0 + \gamma^* J \frac{L_0^2}{4} + f \frac{L_0}{2})$  energy per particle is needed to overcome the viscous drag force  $\gamma^* V/2$ , the potential barrier  $U_0$  and the external load  $f$ . Here  $\gamma^* = B(e^{-AT_h} + e^{-AT_c})$ . In addition, an amount of  $\frac{1}{2}k_B(T_h - T_c)$  energy per cycle is transferred from the hot to the cold heat bath via the kinetic energy at the boundaries of the heat baths. Thus for arbitrary particle crossing through the potential barrier, the amount of heat energy taken from the hot reservoir in a given cycle is given by

$$Q_h = \left( U_0 + \gamma^* J \frac{L_0^2}{4} + f \frac{L_0}{2} + \frac{1}{2}k_B(T_h - T_c) \right). \quad (13)$$

On the other hand the heat given to the cold reservoir is

$$Q_c = \left( U_0 - \gamma^* J \frac{L_0^2}{4} - f \frac{L_0}{2} + \frac{1}{2} k_B (T_h - T_c) \right). \quad (14)$$

If the motor acts as a refrigerator, the net heat flow to the cold heat bath has a magnitude [14]

$$Q_c = \left( U_0 - \gamma^* J \frac{L_0^2}{4} - f \frac{L_0}{2} - \frac{1}{2} k_B (T_h - T_c) \right). \quad (15)$$

In one cycle, the particle does a work of  $W = Q_h - Q_c$  against the load and the viscous friction. Furthermore, the efficiency is given by  $\eta = W/Q_h$ . The performance of the refrigerator is also given as  $P_{ref} = Q_c/W^L$  where  $W^L = fL_0$  is the work done by the load. Here it is also worth mentioning that some motors are not designed to pull loads and in that case an alternative measure for their efficiency depends on the task that each motor performs. For example, some engines may have to achieve high velocity against a frictional drag. This practically suggests that the motor could objectively used to move things a certain distance in a given interval of time. In such motors (where  $f = 0$ ), the useful work is calculated as  $W = (Q_h - Q_c) = \gamma^* J \frac{L_0^2}{2}$ . Once again we want to emphasize that the case where  $A = 0$  corresponds to constant  $\gamma$ . Otherwise when  $A > 0$ ,  $\gamma$  is temperature dependent and in this case we measure work in the unit where  $A$  is taken to be unity for convenience.

We now introduce the dimensionless quantities for energy, length, and time to simplify the model equations. We measure energy in units of  $k_B T_c$  (with  $K_B = 1$ ), hence the load, temperature and barrier height are rescaled as  $\tilde{f} = fL_0/T_c$ ,  $\tilde{T}(x) = T(x)/T_c$ , and  $\tilde{U}_0 = U_0/T_c$  respectively. Similarly we rescale length and time as  $\tilde{x} = x/L_0$  and  $\tilde{t} = t/\beta$  respectively. Here  $\beta = \gamma(x)L_0^2/T_c$  is the relaxation time. For convenience we use  $\tau = T_h/T_c$  to measure the temperature of the hot region. From now on all equations will be expressed in terms of the dimensionless parameters and, hence for brevity we dropped all the bars. We also work in a unit where  $A = 1$  for the non-constant viscous friction.

### III. THE ROLE OF VISCOSITY

Previous studies on Brownian heat engine working due to specially arranged thermal gradient have given us an insight on how the engine thermodynamic features depend on the model parameters. These investigations depicted that the particle attains a unidirectional motions as long as a distinct temperature difference is retained along the bistable potential. In the

presence of a load, the engine exhibits an intriguing dynamics where the magnitude of the load dictates the direction of the particle flow. So far most of the studies assumed the viscosity of the medium to be temperature independent. However, the viscosity of the medium indeed significantly relies on the intensity of the background temperature along the reaction coordinate.

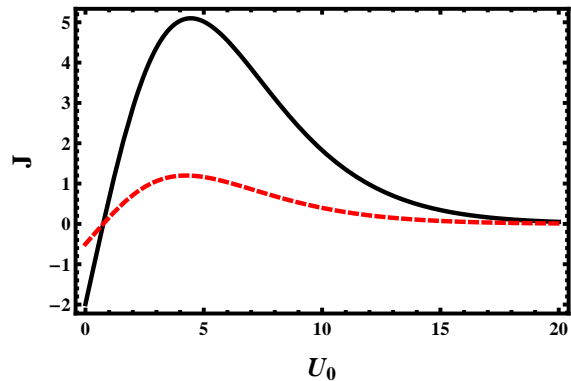


FIG. 2: The current  $J$  as a function of  $U_0$  for the parameter values of  $f = 0.3$ , and  $\tau = 2.0$ . The black solid line stands for temperature dependent viscous friction case while the red line exhibits the current for constant  $\gamma$ .

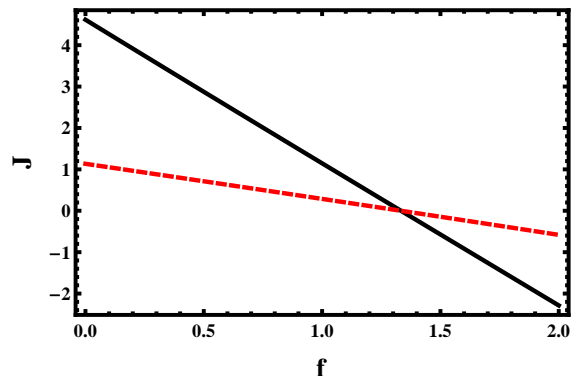


FIG. 3: The current  $J$  as a function of  $f$  for parameter choice  $\tau = 2.0$  and  $U_0 = 2$ . The black solid stands for the current that is evaluated by taking  $\gamma$  to be temperature dependent ( $A = 1$ ) while the red line is plotted by taking  $\gamma$  to be constant ( $A = 0$ ).

In this section we will explore the role of temperature on the performance of the Brownian motor we considered. Here the viscous medium is assumed to vary exponentially with the temperature. As shown in Eq. (6) the steady state current (or equivalently the velocity) is solved exactly. Exploiting Eq. (6), one can see that the mobility of the particles strictly relies on the potential barrier height. When barrier height is too

small, the particle moves sluggishly in the medium. In the limit  $U_0 \rightarrow 0$ ,  $J \rightarrow 0$  depicting that in order to rectify the random Brownian into a directed motion, the presence of a bistable potential is vital. In the high barrier limit, the particle again moves very slowly. Particularly, when  $U_0 \rightarrow \infty$ , the current  $J$  goes to zero as the particle encounters a difficulty in jumping the potential barrier height. To explore the dependence of  $J$  on  $U_0$ , we have shown in Fig. 2 the plot of  $J$  as a function of  $U_0$  for fixed load of  $f = 0.3$  and  $\tau = 2.0$ . The black solid curve in the figure stands for the current evaluated for temperature dependent  $\gamma$  (with  $A = 1$ ) while the red curve shows the constant  $\gamma$  case where  $A = 0$ . The figure clearly indicates that the velocity of the particle is significantly enhanced when the viscous friction of the medium is temperature dependent.

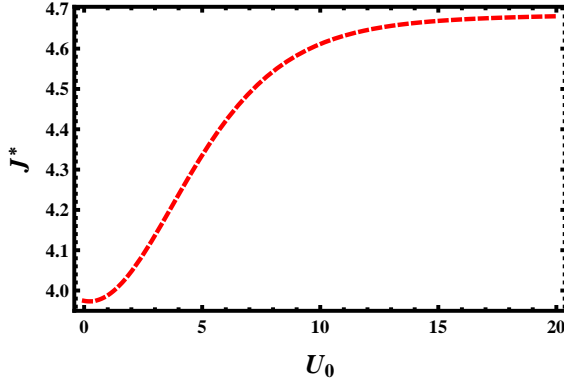


FIG. 4: The ratio of current between temperature dependent  $\gamma$  ( $A = 1$ ) and constant  $\gamma$  ( $A = 0$ ) case as a function of  $U_0$  for the parameter values of  $f = 0.3$  and  $\tau = 2.0$ .

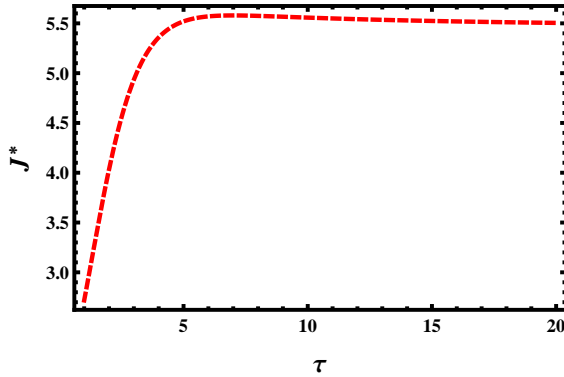


FIG. 5: The ratio of current between temperature dependent  $\gamma$  ( $A = 1$ ) and constant  $\gamma$  ( $A = 0$ ) case as a function of  $\tau$  for the parameter values of  $f = 0.3$ , and  $U_0 = 2.0$ .

In Fig. 3 the current  $J$  is plotted as a function of the load  $f$  for both temperature dependent and constant

$\gamma$  cases. Once again the particle exhibits a higher forward or backward velocity when  $\gamma$  is temperature dependent. The stall force, at which the particle attains a zero velocity, is insensitive to whether  $\gamma$  is temperature dependent or not. Moreover, the direction of particle velocity is dictated by the magnitude of the load. A smaller load lacks the capacity of reverse the current. However, a larger load can reverse the particle motion since it has the capacity of renormalizing the effect of temperature.

The ratio of the two current,  $J^* = J/J^c$ , between the temperature dependent  $\gamma$  ( $A = 1$ ) and constant  $\gamma$  ( $A = 0$ ) cases, as a function of  $U_0$  can be calculated via Eq. (6). In Fig. 4  $J^*$  is shown as a function  $U_0$  for the parameter values of  $f = 0.3$  and  $\tau = 2.0$ . The figure depicts that indeed the velocity for temperature dependent  $\gamma$  case is higher than that of constant  $\gamma$ . Particularly, as the potential barrier increases,  $J^*$  increases considerably. This can be further appreciated by explicitly evaluating  $J^*$  in the limiting cases. For  $f = 0$  case, Eq. (9) reduces to

$$J^* = \frac{2e^{1+\tau}(1+\tau)}{\frac{(-e+e^\tau)U_0}{-1+e^\tau} + \frac{2e^{\tau+U_0}+2e^{1+U_0}\tau-e^\tau(2+U_0)+e(-2\tau+U_0)}{-1+e^{U_0}}} \quad (16)$$

For small barrier height, Eq. (16) converges to

$$J^* \approx \frac{e^{(\tau+1)}}{(e+e^\tau)}. \quad (17)$$

$J^* > 1$  showing that the mobility of the particle is high for  $A = 1$  (temperature dependent  $\gamma$  case). On the other hand for large  $U_0$ ,  $J^*$  is approximated as

$$J^* \approx \frac{e^{(\tau+1)}}{(e\tau+e^\tau)}. \quad (18)$$

Once again  $J^* > 1$  revealing that the velocity of the particle is higher for  $A = 1$  (temperature dependent  $\gamma$  case).

The plot of  $J^*$  as a function of  $\tau$  depicts that  $J^*$  increases with  $\tau$  and attains an optimum value for a certain value of  $\tau$  (see Fig.5). It then decreases when  $\tau$  further increases. For very small  $\tau$ , Eq. (16) leads to  $J^* = \frac{1}{2}e(\tau+1)$ . In the limit  $\tau \rightarrow 1$ ,  $J^* \rightarrow e$ .

All these results indicate that the thermal background temperature undoubtedly affects the strength of the viscosity of the medium and hence this effect cannot be avoided. Particularly an enhanced mobility of the particle is observed when the difference between the hot and cold regions in the background temperatures is big. A similar effect is also observed when the potential barrier is increased.

We now explore the dependence of the efficiency  $\eta$  and the performance of the refrigerator  $P_{ref}$  on the model parameters. To start with we first look at how



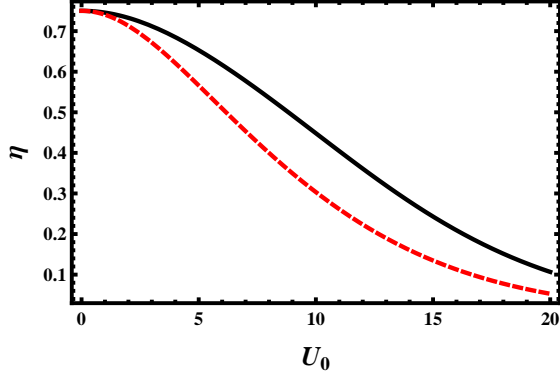


FIG. 6: The efficiency  $\eta$  as a function of  $U_0$  for the parameter values of  $f = 0.0$ , and  $\tau = 4.0$ .

the performance of the engine depends on the barrier height and the load by omitting the heat exchange via kinetic energy. Fig. 6 shows the efficiency  $\eta$  as a function of  $U_0$  for the parameter values of  $f = 0.0$ , and  $\tau = 4.0$ . In this figure, the solid line and dashed lines indicate the  $A = 1$  and  $A = 0$  cases respectively. For both cases, the  $\eta$  decreases from its maximum quasistatic efficiency (Carnot efficiency) when  $U_0$  increases. Far from the quasistatic limit, the temperature dependent medium gives a larger  $\eta$  as compared to the constant  $\gamma$  case.

It is important here to note that the quasistatic limit corresponds to the case where  $J \rightarrow 0$ . The current approaches zero when  $U_0 \rightarrow 0$  for zero external load. In the presence external load, the particle current vanishes when

$$f = 2U_0 \left( \frac{\tau - 1}{\tau + 1} \right). \quad (19)$$

We find that at quasistatic limit (for both  $A = 1$  and  $A = 0$  cases), the efficiency always goes to Carnot efficiency, *i. e.*

$$\eta = 1 - \frac{1}{\tau}. \quad (20)$$

The performance  $P_{ref}$  of the refrigerator is explored as a function of model parameters. At quasistatic limit (regardless of the choice of  $A$ ),  $P_{ref}$  always approaches to the Carnot refrigerator,

$$P_{ref} = \frac{1}{\tau - 1}. \quad (21)$$

In Fig. 7 we have shown  $P_{ref}$  as a function of  $U_0$ . Clearly the figure indicates that the performance gets weaker when the medium friction is temperature dependent. The two operation regions of the engine in parameter space of  $f$  and  $U_0$  is also shown in Fig. 8.

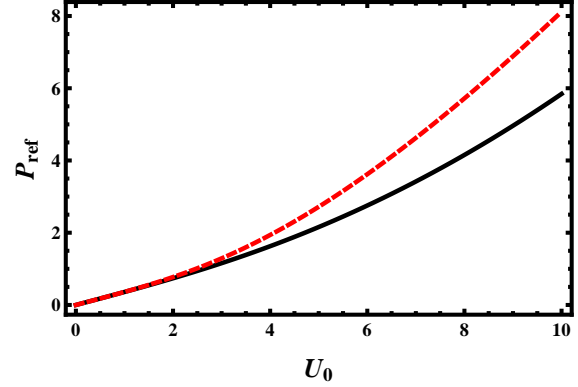


FIG. 7:  $P_{ref}$  as a function of  $U_0$  for the parameter values of  $f = 0.8$ , and  $\tau = 6.0$ . The solid black line stands the current which is plotted by taking  $\gamma$  to be temperature dependent while the red dashed line is plotted by taking  $\gamma$  to be constant.

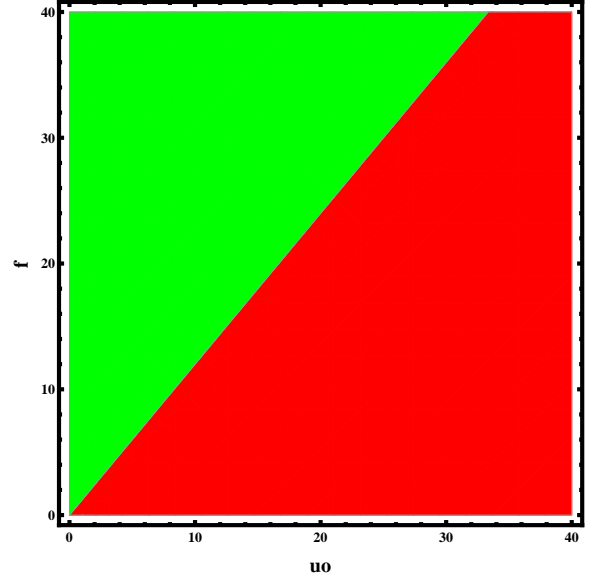


FIG. 8: The two operation region of the engine in parameter space of  $f$  and  $U_0$  for a fixed  $\tau = 2.0$ . In the region that marked red, the model works as a heat engine while in the region that marked in green the model acts as a refrigerator.

In this figure the region marked by red is the region where the model works as a heat engine while green is the region where the model acts as a refrigerator.

We finally examine the thermodynamic property of the engine by including the heat exchange via the kinetic and the potential energies. When the system works as a heat engine, the net flux of the particle is from hot to the cold heat baths. Similarly, when the

engine acts as a refrigerator, the net flow of the particle is from the cold to the hot reservoir. Hence when a particle moves from hot to cold heat bath, in one cycle, an amount of  $\frac{1}{2}k_B(T_h - T_c)$  energy is transferred via kinetic energy.

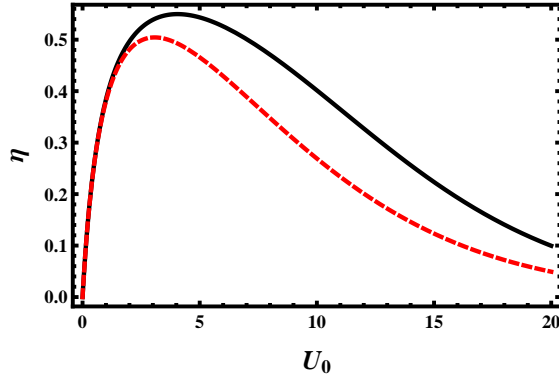


FIG. 9: The efficiency  $\eta$  as a function of  $U_0$  for the parameter values of  $f = 0.0$ , and  $\tau = 4.0$ . The efficiency is plotted by considering the heat exchange via kinetic energy.

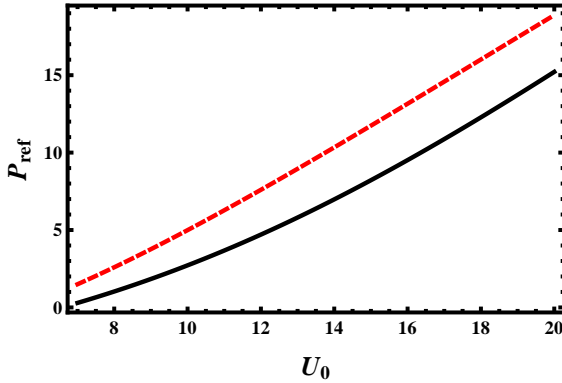


FIG. 10:  $P_{ref}$  as a function of  $U_0$  for the parameter values of  $f = 0.8$ , and  $\tau = 6.0$ . The solid line stands the current which is plotted by taking  $\gamma$  to be temperature dependent while the dashed line is plotted by taking  $\gamma$  to be constant. The energy exchange via kinetic energy is taken into account

When the heat exchange via the kinetic energy is included, Carnot efficiency will not be attained for the model even at the quasistatic limit. This is due to the fact that the heat flow via kinetic energy is irreversible.

We have also examined the dependence of the efficiency in the presence of the kinetic energy heat flow for the model parameters. In the quasistatic limit, the

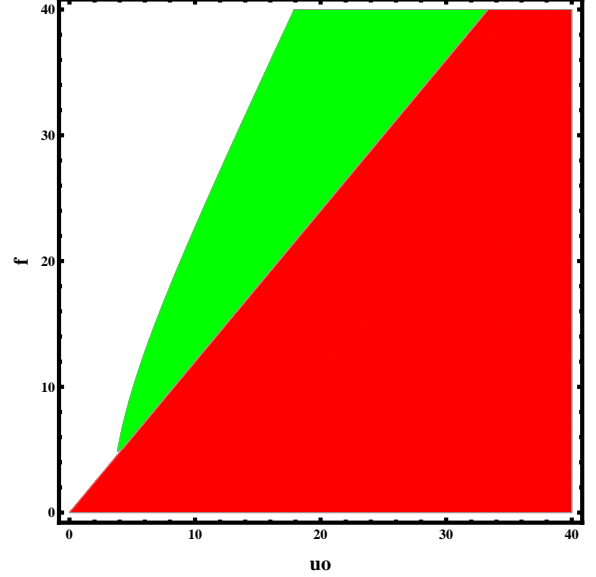


FIG. 11: The phase diagram in parameter space of  $U_0$  and  $f$ . In the region that marked red, the model works as a heat engine while in the region that marked in green the model acts as a refrigerator. On the other hand, in the white region, the model acts neither as a heat engine nor as refrigerator.

steady state efficiency takes a form

$$\eta^{irr} = \frac{\tau - 1}{\tau} \Omega \quad (22)$$

where

$$\Omega = \frac{4U_0}{4U_0\tau - \tau^2 - 1} \quad (23)$$

Here  $0 < \Omega < 1$  reveal that the efficiency can never approaches the Carnot efficiency even at quasistatic. Hence  $\eta^{irr} < \eta$ . At the quasistatic limit, the efficiency remains close to zero for both large and small values of the barrier height  $U_0$  as shown in Fig. 9. However, the efficiency has an optimal value at certain barrier height.

The heat flow via the kinetic energy has also influence on the performance of the refrigerator. Our analytical result show that the coefficient of performance of the refrigerator is always less than the Carnot refrigerator when the engine operates quasistatically. In the quasistatic limit, the steady state  $P_{irr}^{ref}$  converges to

$$P_{irr}^{ref}(t) = \frac{1}{\tau - 1} \Psi \quad (24)$$

where

$$\Psi = \frac{\frac{4U_0}{\tau} - \tau^2 + 1}{4U_0}. \quad (25)$$

Again here  $0 < \Psi < 1$  reveals that Carnot refrigerator is unattainable even at quasistatic limit. When the engine operates at finite time, the system exhibits a higher performance of the refrigerator for constant  $\gamma$  case (see Fig. 10).

A complete picture for the operation regions of the heat engine is obtained by observing the phase diagram in parameter space of  $U_0$  and  $f$  as shown in Fig.11. Again here we use the same color scheme where the red shows the region where the model works as a heat engine while green represents the region for a refrigerator. In the region that marked white, the model acts neither as a heat engine nor as refrigerator

#### IV. SUMMARY AND CONCLUSION

In this work, we study the effect of temperature on the performance of the heat engine as well as on its mobility by considering a viscous friction that has an exponential temperature dependence. Our analysis shows that whether the viscous friction is temperature dependent or not, at quasistatic limit, one always gets

Carnot efficiency and Carnot refrigerator provided that the heat exchange via kinetic energy is neglected. However, when the heat exchange via the kinetic energy is included, both Carnot efficiency and Carnot refrigerator are unattainable even at quasistatic limit. Meanwhile, far from quasistatic limit, the engine exhibits an enhanced performance when the viscous friction is taken to be temperature dependent. Our detail analysis indicates that the thermal background temperature has dual effects as it weakens the strength of the viscous friction.

In conclusion, in this work, we present a pragmatic model system that not only serves as a basic understanding of non-equilibrium physics but also for construction of artificial tiny motors that operate at microscopic or nanoscopic levels. Our study depicts that the role of temperature is twofold. It enhances the performance of the motor directly by assisting the particle to surmount the potential barrier or indirectly by weakening the intensity of the viscous friction.

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